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Controlling unstable steady states using system parameter variation and control duration

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We describe a method to control unstable steady states in high-dimensional flows that have become unstable due to a Hopf bifurcation. The control method is model independent and is accomplished by perturbing an available system parameter. Unlike methods designed for low-dimensional systems, successful control of high-dimensional systems requires the use of the control duration as an additional parameter. This is due to control activated transients that force the system off a low-dimensional attractor. We demonstrate our method using numerical simulation and compare it with the application of pure discrete control.

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I. INTRODUCTION

Recently there has been extensive interest in model-independent control using only small perturbations of an available system parameter. Control methods used for complicated nonlinear systems require accurate and detailed knowledge of a mathematical model, which is often difficult to obtain. Furthermore, perturbations of a system parameter designed to control the system may have the effect of modifying the system's behavior. Control methods that do not require a model and use only small perturbations that do not modify the system behavior can thus be successful where more traditional control methods have failed. In this paper, we devise a model-independent method to control systems that uses not only small parameter fluctuations, but judiciously applies the perturbations for a particular duration of time. We call the time interval during which control is activated by parameter perturbation, *control duration.*

Ott, Grobogi, and Yorke (OGY) [1] designed a method to control unstable steady states and periodic orbits embedded within a chaotic attractor. Since the chaotic attractor is ergodic, the system state will eventually be arbitrarily close to the desired unstable motion. Control then requires only small perturbations of an available parameter to keep the system in this unstable state. The control method is a special case of "pole placement" for linear maps, wherein the system is directed to the stable manifold of the desired state [2]. The OGY method is based upon a map describing the linearized dynamics about the desired state. Information required to construct the map can be obtained from experimental data and does not depend upon the formulation of a mathematical model of the system. Application of the OGY method is particularly suited to low-dimensional attractors where the number of unstable eigenvalues is 1 [3].

One limitation of the OGY control method occurs in high-dimensional systems where the control parameter fluctuations induce transients that hinder the effectiveness of the method. The asymptotic behavior of a high-dimensional system usually lies close to a lower-dimensional attractor, and the design of the OGY control method is based on measurements of this lower-dimensional system. When control is applied by changing a system parameter even a small amount, the resulting transients cause the system state to wander in the high-dimensional space of the system; that is, the dynamic transients leave the low-dimensional attractor. In these situations the OGY method is inappropriate, since the linearization fails. In [3] the authors overcome this difficulty by modifying the original OGY method to take into account the time history of the system as the parameter has been modified; previous states of the system and the control parameter are used to improve the local linear model and determine the proper control perturbation. Multiple independent control parameters have also been used to account for the behavior of the system off the original attractor [4]; for instance, while the first parameter may push the dynamics off the attractor, a second parameter perturbation may push the dynamics back onto the original attractor, so that the effect of the transients is neutralized.

A control method related to OGY called occasional proportional feedback (OPF) has been successful in controlling a number of physical systems [5-8]. In these experiments, a small change is made to a system parameter to correct for deviations of a single system variable from a desired reference point. This single variable is sampled only occasionally, where the sample interval is usually related to the natural period of the system. For these reasons, OPF has been called a one-dimensional version of OGY when the contracting dynamics of the other variables is very strong [9]. However, there are many variables that must be tuned when implementing OPF in a real experiment, and a clear understanding of the role of each is lacking. Of particular interest is the use of OPF to control the steady state in a multimode laser [8], which is a high-dimensional system. In this experiment the steady state has become unstable due to a Hopf bifurcation.

Our goal in this paper is to further understand the relation between OPF and OGY in controlling the steady state behavior of high-dimensional systems. To this end, we derive a one-dimensional version of OGY and successfully apply it to the steady state control of a two-dimensional chemical system. However, when implementing the algorithm in a higher-dimensional system, where the additional dimensions are strongly contracting, the OGY method fails.

We then derive an alternative method to control unstable steady states occurring in a high-dimensional system. In particular, we consider steady states with a complex-conjugate pair of unstable eigenvalues; the number and type of additional eigenvalues is arbitrary except that they must be stable. The control method follows the theme of the OGY method in that we assume that the system is close to the desired steady state, so that only small perturbations are required to maintain control. Furthermore, we assume that the dynamics close to the steady state can be estimated from experimental data and hence no model is needed. However, we consider the system flow over time, in contrast to the OGY method which employs a map constructed from intersections with a Poincare' surface. In addition to considering the system flow, the novel aspect of our algorithm is that we determine an optimal duration of time to apply the control. Given an initial error from the desired steady state, we determine the appropriate control parameter perturbation and how long this perturbation is to be applied, so that the system is directed back to the unstable steady state. By following the flow we are explicitly taking into account any transients that occur as the parameter changes. This is difficult to do in a map approach, as the parameter changes can be made only on iterates of the map. We will demonstrate our method on the Lorenz equations by controlling the nonzero steady state solution after it has become unstable due to a Hopf bifurcation.

In deriving the control algorithm, we consider the general system

$$d\frac{\mathbf{z}}{dt}=\mathbf{F}(\mathbf{z},P) , \tag{1}$$

where $\mathbf{z}$ is an n-dimensional state variable, P is a scaler parameter of the system and will be used as the control variable, and $\mathbf{F}$ is a nonlinear function of the state and control variables. We assume the existence of a steady state solution given by $(\mathbf{z}(P),P)$. We wish to establish control about a particular steady state when $P=\bar{P}$ and $\mathbf{z}(\bar{P})=\bar{\mathbf{z}}$.

To this end we approximate the dynamics about this steady state point as

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}\cdot\mathbf{x}+\mathbf{B}p , \qquad \mathbf{A}=D_z\mathbf{F}(\bar{z},\bar{P}) \text{ and } \mathbf{B}=\frac{d}{dP}\mathbf{F}(\bar{z},\bar{P}) , \tag{2}$$

where $\mathbf{x}$ and p are small deviations ($\mathbf{x}=\mathbf{z}-\bar{\mathbf{z}}\ll 1$ and $p=P-\bar{P}\ll 1$) from the steady state values $\bar{z}$ and $\bar{P}$, respectively. We assume that there is a single complex-conjugate pair of unstable eigenvalues $\sigma_u(p)\pm i\omega(p)$, where $\sigma_u>0$ and $d\sigma_u/dp\neq 0$. This implies the existence of a Hopf bifurcation for some lower value of the parameter P. For a low-dimensional system ($n=2$) these will be the only eigenvalues, while in high-dimensional systems ($n>2$) the only restriction on the additional eigenvalues is that they have a negative real part. In the case of model-independent control, we assume that all the parameters of (2), i.e., $\mathbf{A}$, $\mathbf{B}$, and hence all the eigenvalues and eigenvectors, can be determined by embedding the flow of some real system into an artificial phase space [15].

This paper is organized as follows. In Sec. II we derive a one-dimensional version of OGY and apply it to a simple chemical reaction. In Sec. III we derive a control method good for high-dimensional systems that overcomes the limitations of the method derived in Sec. II, while in Sec. IV we apply our method to an actual system. In Sec. V we summarize the results.

II. CONTROL OF A LOW-DIMENSIONAL SYSTEM

We consider the case $n=2$ in (1) and (2), such that the dynamics occur in the plane. Due to a Hopf bifurcation, the steady state solution is an unstable focus, and errors will spiral away from the steady state. By sampling the system at a specific phase, we can consider the dynamics to be modeled by a monotone map of the form

$$x_{n+1}=ax_n+bp, \quad x_n=\left|\left|x\left[\frac{2\pi}{\omega}n\right]\right|\right| , \tag{3}$$

where x_n represents the present error or distance from the fixed point, and p is the control variable. The growth of errors away from the steady state between samples is measured by the parameter a, which is given by

$$a=e^{2\pi\sigma_u/\omega} . \tag{4}$$

To calculate b we make the observation that the fixed point of the map (3) represents the unstable steady state of the flow (2). We thus require the change in the fixed point with respect to the control variable p to be the same as the change in the steady state with the control variable p, to obtain

$$b=-(\mathbf{A}^{-1}\cdot\mathbf{B})(1-a) . \tag{5}$$

The map is then fully determined in terms of the known parameters of (2). The goal of control is that given the present error x_n, control should eliminate subsequent errors, or $x_{n+1}=0$. The required control-parameter perturbation is obtained by solving (3) so that p is given by

$$p=-\frac{a}{b}x_n . \tag{6}$$

Using these ideas we control the steady state of a simple chemical reaction by considering the Brusselator [11] without diffusion. It is given by

$$\frac{dX}{dt}=Q-(P+1)X+X^2Y , \qquad \frac{dY}{dt}=PX-X^2Y , \tag{7}$$

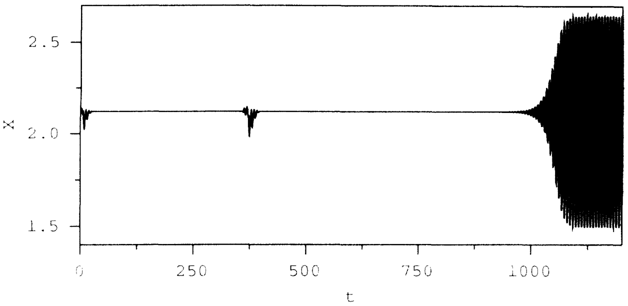

FIG. 1. Control of the Brusselator for $\bar{P}=2.12$ is established after an initial error, and then reestablished after the system is perturbed. Once control is turned off, the system evolves to periodic oscillations.

where we will fix the parameter $Q=1$, and P will be the control variable. The steady state solution $(X,Y)=(Q,P/Q)$ undergoes a Hopf bifurcation when $P=(Q^2+1)$.

We will maintain the system at the steady state obtained when $\bar{P}=2.12$. The deviation of X away from the steady state value, measured when $Y=0$, determines x_n. The control variable P is then modified iteratively according to (6). Figure 1 shows control being established after an initial error, reestablished after we artificially perturb the system, and the evolution to periodic oscillations when control is turned off. Figure 2 shows a detailed view of the control perturbations and the effect on X after the system has been perturbed. Note that the system is constantly being controlled, although the magnitude of the perturbations in P become imperceptibly small.

Because the dynamics is restricted to a plane, this method works successfully as a one-dimensional controller, as predicted in [9]. However, we were unable to apply this method to higher-dimensional systems, such as the Lorenz equations [see (14)]. The three-dimensional Lorenz system is similar in that the nonzero steady state undergoes a Hopf bifurcation, while the stable direction remains strongly attracting. However, when the control

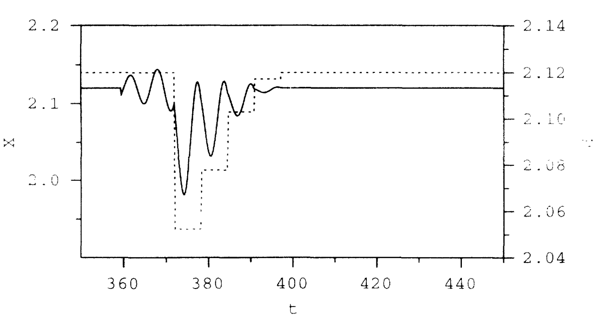

FIG. 2. Detailed view of Fig. 1 when control is being reestablished, where the dotted line is the control variable. Note that the system is allowed to grow freely for approximately two cycles before control is reactivated, after which the fluctuations in X and P become imperceptible.

parameter was modified, the system state shifted off the unstable manifold of the uncontrolled steady state. In the case of the Brusselator, the parameter perturbations shifted only the steady state's location in the unstable manifold; i.e., the two-dimensional plane. Even though the dynamics of the unstable manifold of the Lorenz equations were strongly attracting, the control method failed due to the fact that the whole unstable manifold had shifted. Petrov, Peng, and Showalter [4] overcome this difficulty by shifting the unstable manifold back using an additional control parameter. In Sec. III we present an alternative solution that does not require an additional parameter but does use the control duration.

III. CONTROL ALGORITHM FOR HIGH-DIMENSIONAL SYSTEMS

For high-dimensional systems we again assume a system modeled by (1) and (2). Provided that $\mathbf{A}$ is nonsingular, i.e., $P=\bar{P}$ is not a bifurcation point, then the general solution to (2) is

$$\mathbf{x}(t)=e^{\mathbf{A}t}\cdot[\mathbf{x}(0)+\mathbf{A}^{-1}\cdot\mathbf{B}p]-\mathbf{A}^{-1}\cdot\mathbf{B}p\ . \tag{8}$$

The $n\times n$ matrix $\mathbf{A}$ can be block diagonalized as $\mathbf{A}=\mathbf{S}\cdot\Lambda\cdot\mathbf{S}^{-1}$, where $\mathbf{S}$ is composed of the right eigenvectors $\mathbf{e}_i$, $i=1,\ldots,n$, and $\mathbf{S}^{-1}$ is composed of the left eigenvectors $\mathbf{f}_i$, $i=1,\ldots,n$. We write the matrix of eigenvalues Λ in the form

$$\Lambda=\begin{vmatrix} \sigma_u & \omega & 0 & \cdot & 0 \\ -\omega & \sigma_u & 0 & \cdot & 0 \\ 0 & 0 & \sigma_{s3} & \cdot & 0 \\ \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \sigma_{sn} \end{vmatrix}\ , \tag{9}$$

where $\sigma_u>0$ is the growth rate and ω is the frequency of the unstable modes designated as $\mathbf{e}_1$ and $\mathbf{e}_2$ (similarly, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the corresponding left eigenvectors). In (9) we have shown the case where all other eigenvalues are real such that $\sigma_{si}<0$, $i=3,\ldots,n$. In general, the stable modes may be complex with negative real parts. The control method that we derive is not effected by this modification.

The goal of the control method is that given some initial error $\mathbf{x}(0)$, we determine the parameter variation p and the control duration $T_c=(2\pi q)/\omega$ (q is the unknown), such that

$$\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{0}\ . \tag{10}$$

This is accomplished by forcing the system state to lie entirely within the stable manifold at the end of the control period, i.e.,

$$\mathbf{x}(T_c)=\mathbf{S}\cdot\mathbf{k}\quad\text{where } \mathbf{k}=(0,0,k_3,\ldots,k_n)\ , \tag{11}$$

where k_i are to be determined. Substituting $t=(2\pi q)/\omega$ into (8), we obtain n equations for the n unknowns $p,q,k_3,\ldots,k_n$. We are concerned only with the first two equations that determine p and q. Once these are determined the remaining equations determine only k_i. We allow these to be arbitrary because they specify only the location of the system in the stable manifold. Once the dynamics is on the stable manifold, the system will then evolve toward the steady state, as stated in (10).

Solving the first two equations of (8) for p, we obtain

$$p=\frac{-\lambda(q)\mathbf{f}_1\cdot\mathbf{x}(0)}{d_1[\lambda(q)-\cos(2\pi q)]+d_2\sin(2\pi q)}\ ,$$

$$p=\frac{-\lambda(q)\mathbf{f}_2\cdot\mathbf{x}(0)}{d_2[\lambda(q)-\cos(2\pi q)]-d_1\sin(2\pi q)}\ ,$$

$$\lambda(q)=e^{2\pi\sigma_u q/\omega}\ , \qquad (12)$$

$$d_1=\frac{(\sigma_u\mathbf{f}_1-\omega\mathbf{f}_2)\cdot\mathbf{B}}{\sigma_u^2+\omega^2}\ ,$$

$$d_2=\frac{(\omega\mathbf{f}_1+\sigma_u\mathbf{f}_2)\cdot\mathbf{B}}{\sigma_u^2+\omega^2}\ .$$

Equating the two equations for p in (12) yields the following transcendental equation for q:

$$[(\lambda(q)-\cos(2\pi q)](d_1\mathbf{f}_2-d_2\mathbf{f}_1) +\sin(2\pi q)(d_1\mathbf{f}_1+d_2\mathbf{f}_2)\cdot\mathbf{x}(0)=0\ . \qquad (13)$$

Note that (13) depends upon the relative angle between $\mathbf{x}(0)$ and $\mathbf{f}_1$ and $\mathbf{x}(0)$ and $\mathbf{f}_2$, but not on the magnitude of $\mathbf{x}(0)$. The solution to (13) is multivalued and we take $q\in(0,1)$, so that control is applied for less than one natural period of the system.

To control the system we have the following algorithm: For any given $\mathbf{x}(0)$, (13) is solved numerically to determine q. Next, p is determined using either of the relations in (12). Using these values, $\mathbf{x}(T_c)$ will lie in the stable manifold and eventually decay to 0. In a real system, noise and small errors will require the system to be monitored and control reapplied.

IV. APPLICATION TO THE LORENZ EQUATIONS

We control a higher-dimensional system when $n=3$ using the well-known Lorenz equations [10]. They are given by

$$\frac{dx}{dt}=\sigma(y-x)\ ,$$

$$\frac{dy}{dt}=rx-y-xz\ ,$$

$$\frac{dz}{dt}=xy-bz\ .$$

For the parameter values of $b=\frac{8}{3}$ and $\sigma=10$, the stable steady state solution $(x,y,z)=(\sqrt{b(r-1)},\sqrt{b(r-1)},r-1)$ undergoes a Hopf bifurcation at an approximate value of $r=24.74$. . . . There is a complex-conjugate pair of eigenvalues with positive real part and a real negative eigenvalue, which are functions of r. We will stabilize the steady state when $r=28$, for which we can explicitly calculate σ, ω, $\mathbf{f}_1$, and $\mathbf{f}_2$ as required in (12). Thus p will represent deviations of r from $r=28$, and the control duration is $(2\pi q)/\omega$.

The Lorenz equations have been a popular system to demonstrate the control of chaos. Abed and Wang [12] use the combination of a washout filter and nonlinear control; the former is used to postpone the Hopf bifurcation point, while the nonlinear control is used to stabilize the chaotic transients. Singer, Wang, and Bau [13,14] use negative feedback for steady state control of a thermal convection loop based on the Lorenz equations. In both cases, they use continuous control.

The control algorithm described in Sec. III was implemented by first specifying a control criteria. For example, when the peak value of the oscillations of x (subsequently referred to as Δx) reached a specified maximum value, p and q were determined using (12) and (13) and control applied for an appropriate amount of time. The system is then allowed to evolve until the control criteria is again satisfied. In Fig. 3 we show results when $\Delta x=0.2$; if the control is turned off, the system then becomes chaotic. Note the long time between control perturbations with respect to the natural period of the system. In Fig. 4 the action of a single control correction is shown; the effectiveness of the method is demonstrated by the fact that after the control is turned off, oscillations are imperceptible. Since the control criteria is monitoring the maximum of x, the system is in nearly the same phase in the unstable manifold when p and q are determined, the variations being due to the overshoot of the peak of x from Δx. The parameter q explicitly depends upon this phase, so that the variations around the mean value $q=0.71995$ are very small. This is the reason why p was always positive.

The control method is robust in the presence of noise, as shown in Fig. 5(a), where uniformly distributed random perturbations in the range $(-0.05,0.05)$ were added to the flow variables. There exist regions where bursts in the system require large control perturbations, as shown in Fig. 5(b). An approach used when controlling periodic orbits embedded in a chaotic attractor is to disallow large

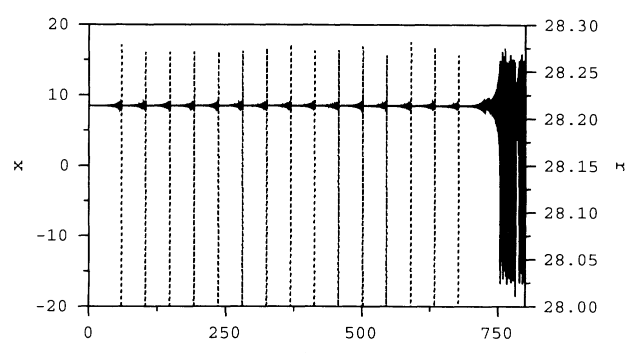

FIG. 3. Control of the positive nonzero steady state of the Lorenz equations. The error between the variable x and the unstable steady state is allowed to grow until the difference is 0.2. Control is then activated (the dotted lines) by determining the appropriate perturbation to r and the duration of time it is to be applied. Once control is turned off the system becomes chaotic.

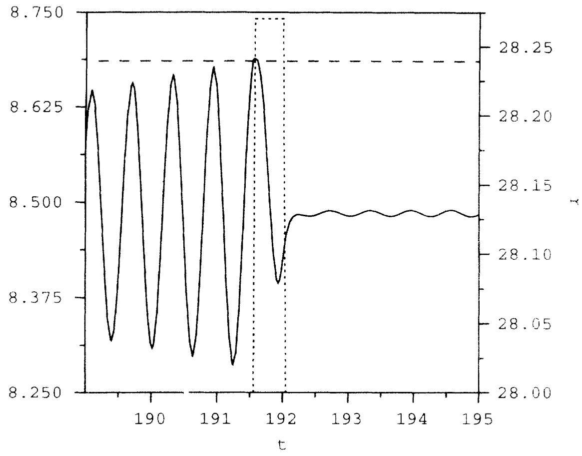

FIG. 4. Detailed view of the effect of control. When $\Delta x > 0.2$ (the peak value of x, indicated by the solid line, is greater than the dashed reference line), control is activated using a small perturbation in the control variable r, indicated by the dotted line. Control is one for less than the period of the oscillations.

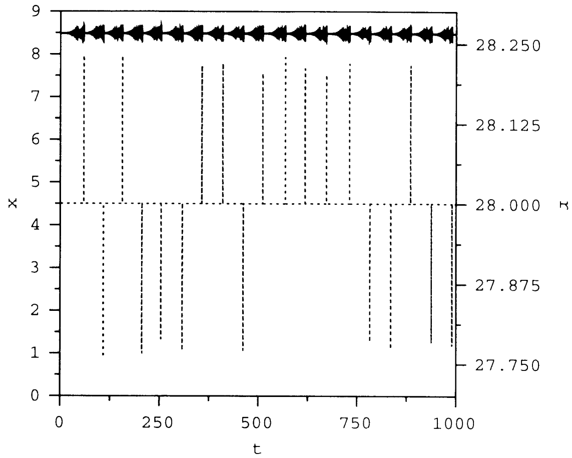

FIG. 6. In contrast to Fig. 3, control is activated when the magnitude of the error *vector* is greater than 0.38. The system is sampled at arbitrary phases in the unstable manifold, so that there is a wide range of values of q [$q \in (0.57, 0.67)$], while the long-time mean of p will approach zero.

control perturbations, and use the chaotic dynamics to return the system to a small neighborhood of the steady state, as was done in [1] and [2]. Control using only small perturbations can then be turned back on. For the Lorenz system, the chaotic flow will not return the system to the nonzero unstable steady state, so that this method cannot be used.

Equations (12) and (13) determine the values of p and q that give optimal control in the sense that the time between the required control perturbations was maximized.

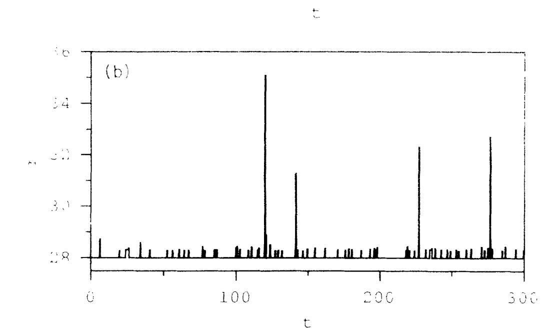

FIG. 5. (a) Control is maintained in the presence of noise, although (b) control corrections occur more often. There are brief bursts of large deviations in the system variable s that require large control perturbations.

We find that the method was quite tolerant of deviations from these values. For instance, in the above example, we determined the appropriate value of p but then fixed q arbitrarily. Control was maintained for deviations from the optimal value of ± 0.2. The effect was that control had to be applied more often because the corrections were less efficient.

Recall that the simulations in Figs. 3-5 implemented control when the maximum of the scaler dependent variable x was greater than a specified value. We called this the control criteria. This is not the only control criteria that may be used. For instance, in Fig. 6 we have specified the maximum magnitude of the error vector $\mathbf{x} = (x, y, z)$. In this case, the phase of the vector $\mathbf{x}$ in the unstable manifold is arbitrary (in the previous case it was fixed, since we always sampled the system on the maximum of x). Since the parameter q depends upon this phase, the range of q will be much greater, allowing p to take on positive and negative values. The mean of p over many control corrections will approach zero. For the time series shown in Fig. 6, $q \in (0.57, 0.67)$, and the mean value of $p = 0.008$.

V. SUMMARY

We have been successful in controlling a steady state in the flow of a high-dimensional system, by using parameter variation and control duration. These considerations were necessary to account for the transients occurring off the attractor when the control parameter was varied. Our approach is impossible when considering the system modeled by a Poincare' map, as the control parameter cannot be modified between the system samples that define the map.

Our results also suggest that OPF may be more than simple a one-dimensional version of the OGY method. In OPF, the control duration is adjusted along with the feedback gain to achieve control. We have seen that our method maintains control, although nonoptimally, when the duration is fixed as is done in the OPF experiments. In either case, successful control depends on using the control duration as an additional parameter.

We remark that the parameters appearing in (12) and (13), which are necessary to determine p and q, can be determined from an experimental time series using embedding techniques to reconstruct the attractor [15]; thus the control method is independent of any specific model. In this case, the control method must be generalized since the state vector is constructed using time delay coordinates [16].

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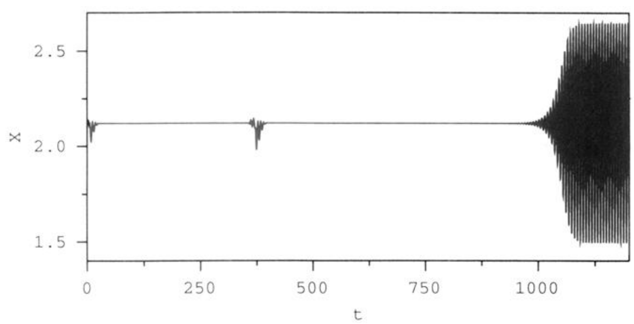

FIG. 1. Control of the Brusselator for $\bar{P}=2.12$ is established after an initial error, and then reestablished after the system is perturbed. Once control is turned off, the system evolves to periodic oscillations.